Dirac structures and dynamical r-matrices

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Abstract

The purpose of this paper is to establish a connection between various subjects such as dynamical r-matrices, Lie bialgebroids, and Lagrangian subalgebras. Our method relies on the theory of Dirac structures developed in [17] [18]. In particular, we give a new method of classifying dynamical r-matrices of simple Lie algebras $\mathfrak{g}$, and prove that dynamical r-matrices are in one-one correspondence with certain Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$.

1 Introduction

Recently, there has been a great deal of interest in the so called *Classical Dynamical Yang-Baxter Equation* (here after *CDYBE*):

$$Alt(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0, \tag{1}$$

where $r(\lambda) : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ is a meromorphic function, and $\mathfrak{g}$ is a complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$. When r is a constant function, Equation (1) reduces to the usual classical Yang-Baxter equation, and therefore a classical r-matrix is a special solution. Assume that r is a solution, and that $r + r^{21} = \epsilon\Omega$, where $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ is the Casimir element corresponding to the Killing form, and ϵ is a constant usually called the coupling constant. Then the skew-symmetric part of r satisfies the following modified CDYBE:

$$Alt(dr) + \frac{1}{2}[r, r] = \frac{\epsilon^2}{4}\,[\Omega^{12}, \Omega^{23}] \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}, \tag{2}$$

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where $[\cdot,\cdot]$ is the Schouten bracket on $\wedge^*\mathfrak{g}$.

In this paper, by a *dynamical r-matrix*, we mean a meomorphic function $r:\mathfrak{h}^*\longrightarrow\mathfrak{g}\wedge\mathfrak{g}$ satisfying:

1. $[h, r(\lambda)] = 0, \quad \forall h \in \mathfrak{h}$, and
2. r satisfies the modified CDYBE (2).

The first assumption is often referred to as the zero weight condition [10]. Here we are mainly interested in dynamical r-matrix with nonzero coupling constant. In this case, by multiplying by a constant, we may always assume that $\epsilon = 2$. In the sequel, we will always make this assumption when referring to a dynamical r-matrix unless otherwise specified.

Classical dynamical r-matrices have appeared in various contexts in mathematical physics, for instance, in Knizhnik-Zamolodchikov-Bernard equation [11], and in the study of integrable systems such as Caloger-Moser systems [2] [5] [6]. A classification of dynamical r-matrices for simple Lie algebras was obtained by Etingof and Varchenko in [10]. An example of such a dynamical r-matrix is

$$r(\lambda) = \sum_{\alpha\in\Delta_+} \coth(<\alpha,\lambda>) E_\alpha \wedge E_{-\alpha},$$

where Δ_+ is the set of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, the E_α and $E_{-\alpha}$'s are root vectors, and $\coth(x) = \frac{e^x+e^{-x}}{e^x-e^{-x}}$ is the hyperbolic cotangent function. Moreover, it is proved that in [10] dynamical r-matrices correspond to Poisson groupoids just as classical r-matrices integrate to Poisson groups in Drinfeld theory [21] [24]. The corresponding Lie bialgebroids, as the infinitesimal invariants, were studied by Bangoura and Kosmann-Schwarzbach [3].

It is well known that there are many ways of producing a classical r-matrix. A natural method is via Lie bialgebras using Manin triples. For instance, for the Lie bialgebra of the standard r-matrix $r_0 = \sum_{\alpha\in\Delta_+} E_\alpha \wedge E_{-\alpha}$, the corresponding Manin triple is $(\mathfrak{g}\oplus\mathfrak{g},\ \mathfrak{g}_1,\ \mathfrak{g}_2)$, where $\mathfrak{g}_1\subset\mathfrak{g}$ is the diagonal while $\mathfrak{g}_2$ is the subalgebra $\{(h+X_+, -h+X_-)|h\in\mathfrak{h}, X_\pm\in n_\pm\}$. Here $n_\pm\subset\mathfrak{g}$ are maximal nilpotent subalgebras. It is thus natural to ask

> **Problem 1**. Does there exist such an analogue for dynamical r-matrices? In particular, what is the double of the Lie bialgebroid corresponding to a dynamical r-matrix?

Recently, Lu has found an interesting connection between dynamical r-matrices and Poisson homogeneous spaces [20]. More precisely, Lu showed that a dynamical r-matrix gives rise to a family of Poisson homogeneous G-spaces G/H parameterized by λ, where G is the Poisson group defined by the standard classical r-matrix r_0 with the same coupling constant (i.e., constant solution of Equation (2)), and H is the subgroup of G having Lie algebra $\mathfrak{h}$. Clearly, the Poisson homogeneous spaces corresponding to different λ, must be related in some way that should reflect the dynamical property of the dynamical r-matrix. This leads to our

> **Problem 2**. Given a family of Poisson homogeneous G-spaces G/H parameterized by λ, what criteria will guarantee that it arises from a dynamical r-matrix?

The infinitesimal object of the Poisson group G is the Lie bialgebra $(\mathfrak{g},\mathfrak{g}^*,r_0)$ generated by the classical r-matrix r_0. According to Drinfeld [9], Poisson homogeneous G-spaces are in one-one correspondence with Lagrangian subalgebras of the double Lie algebra $\mathfrak{d}$, which is isomorphic to the direct sum Lie algebra $\mathfrak{g}\oplus\mathfrak{g}$. So an equivalent formulation of Problem 2 is

Problem 3. Let $W(\lambda) \subset \mathfrak{d}$ be a family of Lagrangian subalgebras. When will this family of Lagrangian subalgebras be induced from a dynamical r-matrix?

In fact Lu showed that these Poisson homogeneous G-spaces include all the Poisson homogeneous G-spaces of the form G/H. This suggests that dynamical r-matrices and Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ should be intrinsically related in some manner. On the other hand, a general classification of Lagrangian subalgebras of $\mathfrak{d}$ has been obtained by Karolinsky [14], which does not seem to have an obvious connection with the work of Etingof and Varchenko [10]. Therefore it is natural to ask

Problem 4. What is the precise relation between dynamical r-matrices and Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$?

Th purpose of this paper is to understand the intrinsic connection between various objects such as dynamical r-matrices, Lagrangian subalgebras, and Lie bialgebroids (see [26]). In particular, our work is motivated by the above questions. Our idea is to use Dirac structure theory developed in [17] [18]. The starting point is a simple Courant algebroid (see Section 3): $(TU \oplus T^*U) \times (\mathfrak{g} \oplus \mathfrak{g})$, which can be considered as an analogue of the direct sum Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ in the algebroid context, where $U \subset \mathfrak{g}^*$ is an open subset. We analyze a class of Dirac structures of this Courant algebroid which are induced from dynamical r-matrices. This study leads to a new method of classification of dynamical r-matrices for simple Lie algebras. One advantage of our approach is that the Cayley transformation, which turns out to be important in classification theory [25], appears quite naturally. We hope that our method may shed new light on the classification scheme of more general dynamical r-matrices [2], and that of dynamical r-matrices for compact Lie algebras. This discussion is the main topic of Section 4. In Section 5, we show that Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ whose intersection with the diagonal are equal to $\mathfrak{h}$, are in one-one correspondence with dynamical r-matrices with zero gauge term. This relates the results of Karolinsky and Lu with that of Etingof and Varchenko in an explicit way. Moreover, we prove that given a point $\mu \in \mathfrak{g}^*$, any such Lagrangian subalgebra W_0 admits a unique extension to a family of Lagrangian subalgebras $W(\lambda)$ with $W(\mu) = W_0$, governed by a dynamical r-matrix. In a certain sense, this is similar to an initial value problem of a first order o.d.e. Section 2 contains some basic facts concerning Lie bialgebroids and Courant algebroids. And Section 3 is devoted to the discussion on the connection between dynamical r-matrices and Lie bialgebroids.

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2 Preliminaries

In this section, we recall some basic facts concerning Lie bialgebroids and Dirac structures.

A Lie bialgebroid is a pair of Lie algebroids $(A,\ A^*)$ satisfying the following compatibility condition (see [22] and [15]):

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y], \quad \forall X, Y \in \Gamma(A), \tag{3}$$

where the differential d_* on $\Gamma(\wedge^* A)$ comes from the Lie algebroid structure on A^*.

Given a Lie algebroid A over P with anchor a and a section $\Lambda \in \Gamma(\wedge^2 A)$, Denote by $\Lambda^{\#}$ the bundle map $A^* \longrightarrow A$ defined by $\Lambda^{\#}(\xi)(\eta) = \Lambda(\xi, \eta), \forall \xi, \eta \in \Gamma(A^*)$. Introduce a bracket on $\Gamma(A^*)$ by

$$[\xi, \eta]_\Lambda = L_{\Lambda^{\#}\xi}\eta - L_{\Lambda^{\#}\eta}\xi - d[\Lambda(\xi, \eta)]. \tag{4}$$

By a_* we denote the composition $a \circ \Lambda^{\#} : A^* \longrightarrow TP$.

Theorem 2.1 *A^* with the bracket and anchor a_* above becomes a Lie algebroid iff*

$$L_X[\Lambda, \Lambda] = [X, [\Lambda, \Lambda]] = 0, \quad \forall X \in \Gamma(A). \tag{5}$$

Proof. In [19], we proved this result with one more condition: $a \circ [\Lambda, \Lambda]^{\#} = 0$, which is equivalent to $[f, [\Lambda, \Lambda]] = 0$, $\forall f \in C^\infty(P)$. But in fact this last condition is a consequence of Equation (5). To see this, by replacing X with fX in Equation (5), one obtains $[fX, [\Lambda, \Lambda]] = 0$. It thus follows that $X \wedge [f, [\Lambda, \Lambda]] = 0$, $\forall X \in \Gamma(A)$, which implies that $[f, [\Lambda, \Lambda]] = 0$.

□

In this case, the induced differential $d_* : \Gamma(A) \longrightarrow \Gamma(\wedge^2 A)$ is simply given by $d_* X = [\Lambda, X]$, $\forall X \in \Gamma(A)$. Thus the compatibility condition, Equation (3), is satisfied automatically. So (A, A^*) is a Lie bialgebroid, called *coboundary Lie bialgebroid*. Λ is also called an r-matrix by abuse of notations. When P reduces to a point, i.e., A is a Lie algebra, Equation (5) is equivalent to that $[\Lambda, \Lambda]$ is *ad*-invariant, i.e, Λ is a classical r-matrix in the ordinary sense. On the other hand, when A is the tangent bundle TP with the standard Lie algebroid structure, Equation (5) is equivalent to that $[\Lambda, \Lambda] = 0$, i.e., Λ is a Poisson tensor.

Given a Lie bialgebroid (A, A^*) over the base P, with anchors a and a_* respectively, let E denote their vector bundle direct sum: $E = A \oplus A^*$. On E, there exists a natural non-degenerate symmetric bilinear form:

$$(X_1 + \xi_1, X_2 + \xi_2) = \frac{1}{2}(\langle \xi_1, X_2 \rangle + \langle \xi_2, X_1 \rangle). \tag{6}$$

In [17], we introduced a bracket on $\Gamma(E)$, called *Courant bracket*:

$$\begin{array}{rcl} [e_1, e_2] & = & \{[X_1, X_2] + L_{\xi_1} X_2 - L_{\xi_2} X_1 - \frac{1}{2} d_*(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)\} \\ & & + \{[\xi_1, \xi_2] + L_{X_1} \xi_2 - L_{X_2} \xi_1 + \frac{1}{2} d(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)\}, \end{array} \tag{7}$$

where $e_1 = X_1 + \xi_1$ and $e_2 = X_2 + \xi_2$. Let $\rho : E \longrightarrow TP$ be the bundle map $\rho = a + a_*$. That is,

$$\rho(X + \xi) = a(X) + a_*(\xi), \quad \forall X \in \Gamma(A) \text{ and } \xi \in \Gamma(A^*). \tag{8}$$

For a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, the bracket (7) reduces to the well known Lie bracket on the double $\mathfrak{g} \oplus \mathfrak{g}^*$. On the other hand, if A is the tangent bundle Lie algebroid TM and $A^* = T^*M$ with zero bracket, then Equation (7) takes the form:

$$[X_1 + \xi_1, X_2 + \xi_2] = [X_1, X_2] + \{L_{X_1} \xi_2 - L_{X_2} \xi_1 + \frac{1}{2} d(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)\}. \tag{9}$$

This is the bracket first introduced by Courant [7]. In general, E together with this bracket and the bundle map ρ satisfies certain properties as outlined in the following:

Theorem 2.2 *[17] Given a Lie bialgebroid (A, A^*), let $E = A \oplus A^*$. Then E, together with the non-degenerate symmetric bilinear form $(\cdot,\cdot)$, the skew-symmetric bracket $[\cdot,\cdot]$ on $\Gamma(E)$ and the bundle map $\rho : E \longrightarrow TP$ as introduced above, satisfies the following properties:*

1. *For any $e_1, e_2, e_3 \in \Gamma(E)$, $[[e_1, e_2], e_3] + c.p. = \mathfrak{D}T(e_1, e_2, e_3)$;*
2. *for any $e_1, e_2 \in \Gamma(E)$, $\rho[e_1, e_2] = [\rho e_1, \rho e_2]$;*
3. *for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(P)$, $[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)\mathfrak{D}f$;*
4. *$\rho \circ \mathfrak{D} = 0$, i.e., for any $f, g \in C^\infty(P)$, $(\mathfrak{D}f, \mathfrak{D}g) = 0$;*
5. *for any $e, h_1, h_2 \in \Gamma(E)$, $\rho(e)(h_1, h_2) = ([e, h_1] + \mathfrak{D}(e, h_1), h_2) + (h_1, [e, h_2] + \mathfrak{D}(e, h_2))$,*

   where
$$T(e_1, e_2, e_3) = \frac{1}{3}([e_1, e_2], e_3) + c.p., \tag{10}$$

and $\mathfrak{D} : C^\infty(P) \longrightarrow \Gamma(E)$ is the map $\mathfrak{D} = d_* + d$.

E is called the double of the Lie bialgebroid (A, A^*). In general, a vector bundle E equipped with the above structures is called a *Courant algebroid* [17].

In this paper, we are mainly interested in a special gauge Lie algebroid $A = TM \times \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra. Clearly A is a Lie algebroid over M with anchor being the projection $p : A \longrightarrow TM$. As for the bracket, note that any section of A can always be written as the sum of a vector field and a $\mathfrak{g}$-valued function on M. The bracket of such two sections is given by:

$$[X + \xi, Y + \eta] = [X, Y] + [\xi, \eta] + L_X\eta - L_Y\xi, \;\; X, Y \in \chi(M), \;\xi, \eta \in C^\infty(M, \mathfrak{g}), \tag{11}$$

where the bracket of two vector fields is the usual bracket and the bracket $[\xi, \eta]$ is the pointwise bracket.

Let $r \in \wedge^2\mathfrak{g}$, which can be considered as a constant section of $\wedge^2 A$. Then

Proposition 2.3 *(A, A^*, r) is a coboundary Lie bialgebroid iff $[r, r]$ is ad-invariant, i.e., iff $(\mathfrak{g}, \mathfrak{g}^*, r)$ is a coboundary Lie bialgebra.*

In this case, the bracket for sections of $A^*(\cong T^*M \times \mathfrak{g}^*)$ is given by

$$[\alpha + \xi, \beta + \eta] = [\xi, \eta], \;\; \alpha, \beta \in \Omega^1(M), \;\forall \xi, \eta \in C^\infty(M, \mathfrak{g}^*), \tag{12}$$

where the right hand side bracket is pointwise bracket on $\mathfrak{g}^*$. The corresponding double is the vector bundle
$$E = A \oplus A^* \cong (TM \oplus T^*M) \times (\mathfrak{g} \oplus \mathfrak{g}^*),$$
where the Courant bracket can be described quite simply. On the subbundle $TM \oplus T^*M$, the bracket is just Courant's original bracket: Equation (9), while for two elements of the double Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ considered as constant sections of E, the bracket is pointwise bracket. One should however note that the subbundle $M \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ is not closed under the Courant bracket (7), since the third property in Theorem 2.2 implies that

$$[fe_1, ge_2] = (fdg - gdf)(e_1, e_2) + fg[e_1, e_2], \;\forall f, g \in C^\infty(M), \forall e_1, e_2 \in \mathfrak{g} \oplus \mathfrak{g}^*, \tag{13}$$

where $fdg - gdf \in \Omega^1(M)$. On the other hand, for $X + \alpha \in \Gamma(TM \oplus T^*M)$, $f \in C^\infty(M)$ and $e \in \mathfrak{g} \oplus \mathfrak{g}^*$, we have

$$[X + \alpha, fe] = L_X(fe) = (Xf)e. \tag{14}$$

These formulas will be needed later on in Section 4.

Given a Courant algebroid E, a *Dirac structure* is a subbundle $L \subset E$ which is maximally isotropic with respect to the symmetric bilinear form $(\cdot,\cdot)$ and is integrable in the sense that $\Gamma(L)$ is closed under the bracket $[\cdot,\cdot]$. There are two important classes of Dirac structures studied in [17]. One is the Dirac structures induced by Hamiltonian operators, and the other is the so called null Dirac structures. Let us briefly recall their definitions below.

Let $H \in \Gamma(\wedge^2 A)$ and denote $H^\# : A^* \longrightarrow A$ the induced bundle map. Then the graph of $H^\#$,

$$\Gamma_H = \{H^\#\xi + \xi | \forall \xi \in A^*\},$$

defines a maximal isotropic subbundle of $A \oplus A^*$. Γ_H is a Dirac subbundle iff H satisfies the Maurer-Cartan type equation:

$$d_* H + \frac{1}{2}[H, H] = 0. \tag{15}$$

In this case we call H a *Hamiltonian operator*. Another interesting class of Dirac structures is the so called *null* Dirac structures, which can be characterized as follows. Let $D \subseteq A$ be a subbundle, and $D^\perp \subseteq A^*$ its conormal subbundle. Consider $L = D \oplus D^\perp \subset A \oplus A^*$. Then L is a Dirac structure iff D and $D^\perp$ are Lie subalgebroids of A and A^*, respectively . In this case L is called a null Dirac structure.

A more general construction of Dirac structures is via the so called characteristic pairs [16]. Let $D \subseteq A$ be a subbundle and $H \in \Gamma(\wedge^2 A)$. Define

$$L = \{\, X + H^\#\xi + \xi \,|\, \forall X \in D, \xi \in D^\perp\} = D \oplus graph(H^\#|_{D^\perp}), \tag{16}$$

where $D^\perp \subseteq A^*$ is the conormal subbundle of D. Clearly, L is a maximal isotropic subbundle of $A \oplus A^*$. The pair (D, H) is called a *characteristic pair* of L.

Conversely, any maximal isotropic subbundle $L \subset A$ such that $L \cap A$ is of constant rank can always be described by such a characteristic pair. Note that two characteristic pairs (D_1, H_1) and (D_2, H_2) define the same subbundle L by Equation (16) iff

$$D_1 = D_2, \quad \text{and} \quad pr(H_1) = pr(H_2), \; i.e., \; H_1 - H_2 \equiv 0 (mod\, D),$$

where pr denotes the projection $A \longrightarrow A/D$ and its induced map $\Gamma(\wedge^* A) \longrightarrow \Gamma(\wedge^* A/D)$. In the above equation as well as in the sequel, a section $\Omega \in \Gamma(\wedge^* A)$ is said equal to zero module D, denoted as $\Omega \equiv 0(mod\, D)$, if its projection under pr vanishes in $\Gamma(\wedge^* A/D)$. Even though L is related only to $pr(H) \in \Gamma(\wedge^2 A/D)$ instead of H itself, it is still convenient to characterize the integrability conditions of L in terms of H, since sections of $\wedge^* A$ admit nice operations such as the exterior derivative and the Schouten bracket.

Theorem 2.4 *([16]) Let (A, A^*) be a Lie bialgebroid, $L \subset A \oplus A^*$ a maximal isotropic subbundle defined by a characteristic pair (D, H) as in Equation (16). Then L is a Dirac structure iff the following three conditions hold:*

1. *$D \subseteq A$ is a Lie subalgebroid.*

2. H satisfies the Maurer-Cartan type equation (mod D):

$$d_* H + \frac{1}{2}[H, H] \equiv 0, (mod D). \tag{17}$$

3. $\Gamma(D^{\perp})$ is closed under the bracket $[\cdot,\cdot] + [\cdot,\cdot]_H$, where $[\cdot,\cdot]_H$ is given by Equation (4). I.e.,

$$[\xi, \eta] + [\xi, \eta]_H \in \Gamma(D^{\perp}), \qquad \forall \xi, \eta \in \Gamma(D^{\perp}). \tag{18}$$

Dirac structures are important in the construction of Lie bialgebroids and Poisson homogeneous spaces. For details, readers may consult the references [17] and [18].

Finally, note that we may also work over $\mathbb{C}$ when M is a complex manifold. In this case, we just need to replace smooth functions by holomorphic functions, and smooth sections by holomorphic sections etc., and all the results above will also hold. In the sequel, we will mainly work with complex Lie algebroids. Even though one normally works with sheaf of local sections when dealing with complex Lie algebroids since there may not exist many global sections. However, in the case below, we can still avoid using sheaf since we are working on an open subset U of $\mathbb{C}^n$.

3 Twists of the standard r-matrix

Dynamical r-matrices have appeared in various contexts [2] [10] [11] [20]. In this section, we will show how a dynamical r-matrix arises naturally as a twist of the standard classical r-matrix in the category of Lie bialgebroids.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a fixed Cartan subalgbra $\mathfrak{h}$ and a root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-, \tag{19}$$

where $\mathfrak{n}_{\pm} = \sum_{\alpha \in \Delta_{\pm}} \mathfrak{g}_{\alpha}$. Let $< \cdot, \cdot >$ denote the Killing form on $\mathfrak{g}$ and $E_{\alpha} \in \mathfrak{g}_{\alpha}$ such that $< E_{\alpha}, E_{-\alpha} >= 1$. Then the standard classical r-matrix r_0 takes the form:

$$r_0 = \sum_{\alpha \in \Delta_+} E_{\alpha} \wedge E_{-\alpha}. \tag{20}$$

Let $h_{\alpha} = [E_{\alpha}, E_{-\alpha}] \in \mathfrak{h}$ for $\alpha \in \Delta_+$ and $h_i = h_{\alpha_i}$ for simple roots α_i, $i = 1, \cdots n$. Then $\{h_1, \cdots, h_n\}$ forms a basis of $\mathfrak{h}$. Let $\{h_1^*, \cdots, h_n^*\}$ be its dual basis, which in turn induces a coordinate system $(\lambda_1, \cdots, \lambda_n)$ of $\mathfrak{h}^*$, i.e., $\lambda = \sum \lambda_i h_i^*$, $\forall \lambda \in \mathfrak{g}^*$.

Now let $U \subset \mathfrak{h}^*$ be a connected open subset. Consider the gauge Lie algebroid:

$$A = TU \times \mathfrak{g} \cong U \times (\mathfrak{h}^* \oplus \mathfrak{g}). \tag{21}$$

Set

$$\theta = \sum_{i=1}^{n} h_i \wedge \frac{\partial}{\partial \lambda_i}. \tag{22}$$

Clearly θ can be considered as a constant section of $\wedge^2 A$. Equip $A^* \cong T^*U \times \mathfrak{g}^*$ with the product Lie algebroid, where T^*U is the trivial Lie algebroid and $\mathfrak{g}^*$ is the dual Lie algebra induced by r_0. Then (A, A^*, r_0) is a coboundary Lie bialgebroid according to Proposition 2.3.

Theorem 3.1 *Let $\tau : U \longrightarrow \wedge^2\mathfrak{g}$ be a holomorphic functions considered as a section of $\wedge^2 A$. Then $\theta + \tau$ is a Hamiltonian operator of the Lie bialgebroid (A, A^*, r_0) iff $r = r_0 + \tau$, the twist of r_0 by τ, is a dynamical r-matrix.*

Proof. $\theta+\tau$ is a Hamiltonian operator iff it satisfies the Maurer-Cartan type equation (see Equation (15)):

$$d_*(\theta + \tau) + \frac{1}{2}[\theta + \tau, \theta + \tau] = 0. \tag{23}$$

By definition, $d_*(\theta+\tau) = [r_0, \theta+\tau]$. Since r_0 is $\mathfrak{h}$-invariant and independent of λ, we have $[r_0, \theta] = 0$. It is also easy to see that $[\theta, \theta] = 0$, and $[\theta, \tau] = \sum(h_i \wedge \frac{\partial \tau}{\partial \lambda_i} + [h_i, \tau] \wedge \frac{\partial}{\partial \lambda_i})$. Thus Equation (23) becomes:

$$\begin{aligned} -\sum [h_i, \tau] \wedge \tfrac{\partial}{\partial \lambda_i} &= (\sum h_i \wedge \tfrac{\partial \tau}{\partial \lambda_i}) + [r_0, \tau] + \tfrac{1}{2}[\tau, \tau] \\ &= (\sum h_i \wedge \tfrac{\partial (r_0+\tau)}{\partial \lambda_i}) + \tfrac{1}{2}[r_0 + \tau, r_0 + \tau] - \tfrac{1}{2}[r_0, r_0] \\ &= Alt(dr) + \tfrac{1}{2}[r, r] - \tfrac{1}{2}[r_0, r_0]. \end{aligned} \tag{24}$$

Now the left side of Equation (24) belongs to $\Gamma(\mathfrak{g}\wedge\mathfrak{g}\wedge TU)$, whereas the right hand side is a section of the subbundle $\wedge^3(U \times \mathfrak{g})$. Thus both sides have to be zero identically. This implies that $[h_i, \tau] = 0$, $\forall i$, i.e., τ is $\mathfrak{h}$-invariant, and r satisfies the modified CDYBE (2) since $\frac{1}{2}[r_0, r_0] = [\Omega^{12}, \Omega^{23}]$.

□

Now assume that $r = r_0 + \tau$ is a dynamical r-matrix. Therefore $\theta + \tau$ is a Hamiltonian operator so that its graph $\Gamma_{\theta+\tau}$ is a Dirac structure of the double of (A, A^*, r_0). Clearly, $\Gamma_{\theta+\tau}$ is transversal to A, so $(A, \Gamma_{\theta+T})$ is a Lie bialgebroid according to Theorem 2.6 in [17]. In fact, it is simple to see that the Lie algebroid $\Gamma_{\theta+\tau}$ is isomorphic to A^* with a twisted bracket defined by the new r-matrix $\Lambda := \theta + \tau + r_0 = \theta + r$, so (A, A^*, Λ) is also a coboundary Lie bialgebroid. Thus, we have proved the following result of Bangoura and Kosmann-Schwarzbach [3]:

Corollary 3.2 *[3] Let $r(\lambda) : U \longrightarrow \wedge^2\mathfrak{g}$ be a holomorphic function. Then $\Lambda = \theta + r(\lambda) \in \Gamma(\wedge^2 A)$ defines a coboundary Lie bialgebroid iff $r(\lambda)$ is a dynamical r-matrix.*

It is not difficult to see that this Lie bialgebroid is the Lie bialgebroid corresponding to the dynamical Poisson groupoid constructed by Etingof and Varchenko [10]. The following conclusion follows immediately from the construction.

Theorem 3.3 *Let $r(\lambda)$ be a dynamical r-matrix, and $\Lambda = \theta + r(\lambda)$ the twisted r-matrix. Then, as a Courant algebroid, the double of the coboundary Lie bialgebroid (A, A^*, Λ) is isomorphic to the double of the untwisted Lie bialgebroid (A, A^*, r_0).*

It is simple to see that a function $\tau : U \longrightarrow \wedge^2\mathfrak{g}$ is $\mathfrak{h}$-invariant iff it can be splitted into two terms: $\tau = \omega + \tau_0$, where

$$\omega = \sum_{ij} \omega^{ij}(\lambda) h_i \wedge h_j, \quad \text{and} \quad \tau_0 = \sum_{\alpha \in \Delta_+} \tau_\alpha(\lambda) E_\alpha \wedge E_{-\alpha}. \tag{25}$$

Proposition 3.4 *Let τ be given as above. Then $\theta + \tau$ is a Hamiltonian operator iff*

1. *τ_0 is a Hamiltonian operator; and*

2. ω is a closed 2-form on U.

Proof. The Maurer-Cartan equation for $\theta+\tau_0+\omega$ takes the form:

$$\begin{aligned} 0 &= d_*(\theta+\tau_0+\omega)+\frac{1}{2}[\theta+\tau_0+\omega,\theta+\tau_0+\omega] \\ &= d_*(\theta+\tau_0)+\frac{1}{2}[\theta+\tau_0,\theta+\tau_0]+[\theta,\omega]. \end{aligned}$$

Note that, on the right hand side of the equation, the only term in $\wedge^3\mathfrak{h}$ is

$$[\theta,\omega]=\sum h_i\wedge\frac{\partial\omega}{\partial\lambda_i}=d\omega.$$

So the equation holds iff

$$d_*(\theta+\tau_0)+\frac{1}{2}[\theta+\tau_0,\theta+\tau_0]=0, \qquad and \qquad d\omega=0.$$

Thus the proposition is proved.

$\square$

In the terminology of Etingof and Varchenko, τ and τ_0 are called gauge equivalent, and ω is a gauge term. In fact, for most purposes we may assume that $\omega=0$.

Finally, note that for any fixed $\lambda\in U$, $\tau(\lambda)\in\wedge^2\mathfrak{g}$ is generally not a Hamiltonian operator for the Lie bialgebra $(\mathfrak{g},\mathfrak{g}^*,r_0)$. In fact, it is easy to see that $r=r_0+\tau$ is a dynamical r- matrix iff

$$[r_0,\tau]+\frac{1}{2}[\tau,\tau]+Alt(d\tau)=0. \tag{26}$$

Thus,

$$\begin{aligned} d_*\tau(\lambda)+\frac{1}{2}[\tau(\lambda),\tau(\lambda)] &= [r_0,\tau(\lambda)]+\frac{1}{2}[\tau(\lambda),\tau(\lambda)] \\ &= ([r_0,\tau]+\frac{1}{2}[\tau,\tau])(\lambda) \\ &= -Alt(d\tau)(\lambda). \end{aligned}$$

So $\tau(\lambda)$ is a Hamiltonian operator iff λ is a critical point of τ (we will see in Section 4 that this is equivalent to $\tau\equiv 0$ on U). Hence $-Alt(d\tau)(\lambda)$ measures the failure of the graph of $\tau(\lambda)^{\#}:\mathfrak{g}^*\longrightarrow\mathfrak{g}$ being a Lagrangian subalgebra. In terms of Drinfel'd [8], $\tau(\lambda)$ is a family of twists, which defines a family of quasi-Lie bialgebras $(\mathfrak{g},\delta(\lambda),\phi(\lambda))$. Here $\delta(\lambda):\mathfrak{g}\longrightarrow\wedge^2\mathfrak{g}$ is given by $\delta(\lambda)(x)=[r_0+\tau(\lambda),x]$ and $\phi(\lambda)=-Alt(d\tau)(\lambda)\in\wedge^3\mathfrak{g}$. This family of quasi Lie bialgebras is the classical limit of the quasi-Hopf algebras studied by Fronsdal [12], Arnaudon et. al. [1] and Jimbo et. al. [13] connected with quantum dynamical R-matrices (see also [27]).

4 Construction of Dirac structures

In the previous section, we have already established a simple connection between dynamical r-matrices and Dirac structures. The purpose of this section is to give an explicit construction of those Dirac structures.

As in Section 3, assume that $\mathfrak{g}$ is a simple Lie algebra with Killing form $<\cdot,\cdot>$, and $r_0 = \sum_{\alpha\in\Delta_+} E_\alpha \wedge E_{-\alpha}$ is the standard r-matrix. By identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ using the Killing form, the bracket on $\mathfrak{g}^*$ is given by:

$$[X,Y]_R = [RX,Y] + [X,RY], \qquad \forall X,Y \in \mathfrak{g},$$

where $R = \pi_+ - \pi_-$, and $\pi_\pm : \mathfrak{g} \longrightarrow \mathfrak{n}_\pm$ are the natural projections with respect to the Gauss decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ as in Equation (19). It is well-known that the double of the Lie bialgebra $(\mathfrak{g},\mathfrak{g}^*)$ can be identified with the direct sum Lie algebra $\mathfrak{d} = \mathfrak{g}\oplus\mathfrak{g}$, while the corresponding invariant non-degenerate bilinear form is:

$$((X_1,Y_1),\,(X_2,Y_2)) \;=\; \frac{1}{2}(<Y_1,Y_2> \;-\; <X_1,X_2>),\;\; \forall X_1,X_2,Y_1,Y_2 \in \mathfrak{g}.$$

Here $\mathfrak{g}$ is identified with the diagonal, while $\mathfrak{g}^*$ is identified with the subalgebra:

$$\{(X_- + h, X_+ - h)|\; \forall X_\pm \in \mathfrak{n}_\pm, h \in \mathfrak{h}\}.$$

Thus the corresponding Courant algebroid, as the double of the Lie bialgebroid (A, A^*, r_0), is a trivial vector bundle, which can be expressed as:

$$E = A \oplus A^* \cong (TU \oplus T^*U) \times \mathfrak{d} \cong U \times (\mathfrak{h}\oplus\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{g}).$$

Consequently, a section of E can be considered as a vector-valued function on U with value in $\mathfrak{h}\oplus\mathfrak{h}\oplus\mathfrak{g}\oplus\mathfrak{g}$, which is denoted by $(\xi(\lambda),\,\eta(\lambda)\,;\,X(\lambda),\,Y(\lambda))$. Here $\xi(\lambda),\eta(\lambda)$ are $\mathfrak{h}$-valued functions on U, and $X(\lambda), Y(\lambda)$ are $\mathfrak{g}$-valued functions on U. The inner-product $(\cdot,\cdot)$ on E is given by

$$\begin{array}{rcl}((\xi_1,\eta_1;X_1,Y_1),(\xi_2,\eta_2;X_2,Y_2)) &=& \frac{1}{2}(<\xi_1,\eta_2> + <\eta_1,\xi_2>) \\ && +\frac{1}{4}(<Y_1,Y_2> - <X_1,X_2>).\end{array} \tag{27}$$

Then as subbundles of E, A and A^* are given by

$$A \;\cong\; U\times\{(k,\,0\,;\,X,\,X)\mid \forall k\in\mathfrak{h}, X\in\mathfrak{g}\}, \qquad \text{and} \tag{28}$$
$$A^* \;\cong\; U\times\{(0,\,h\,;\,X_- + k,\,X_+ - k)\mid \forall h,k\in\mathfrak{h}, X_\pm\in\mathfrak{n}_\pm\}. \tag{29}$$

As for the bracket of $\Gamma(E)$, it admits a simple form for constant sections:

$$[(\xi_1,\,\eta_1\,;\,X_1,\,Y_1),\,(\xi_2,\,\eta_2\,;\,X_2,\,Y_2)] \;=\; (0,\,0\,;\,[X_1,X_2],\,[Y_1,Y_2]). \tag{30}$$

For general sections, the formula is much involved. The following are two special cases corresponding to Equations (13) and (14), which are needed in the future:

$$[(0,0;0,fX),(0,\,0\,;\,0,\,gY)] = (0,\,\frac{1}{4}(gdf - fdg) <X,Y>;\,0,\,fg[X,Y]), \tag{31}$$

and

$$[(h_i^*, h_j; 0, 0), (0,\, 0\,;\, fX,\, gY)] \;=\; (0,\, 0\,;\, \frac{\partial f}{\partial \lambda_i}X,\, \frac{\partial g}{\partial \lambda_i}Y), \quad \forall f, g \in C^\infty(U),\, X, Y \in \mathfrak{g}. \tag{32}$$

Next we need to describe the graph of $\theta^\# + \tau^\# : A^* \longrightarrow A$. For simplicity we assume that τ is given by Equation (25) with $\omega = 0$. Set

$$d = \{(k,\, h\,;\, h+k,\, h-k) \;\mid\; \forall h, k \in \mathfrak{h}\,\} \subset \mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{g} \oplus \mathfrak{g}.$$

And for each $\lambda \in U$, define

$$B(\lambda) = \{(0,\, 0\,;\, (r^\#(\lambda) - 1)X,\, (r^\#(\lambda) + 1)X) \,|\, \forall X \in \mathfrak{n}_\pm\,\}, \tag{33}$$

where $r(\lambda) = r_0 + \tau(\lambda)$ as in Theorem 3.1.

Lemma 4.1 *As a subbundle of E, the graph of $\theta^\# + \tau^\# : A^* \longrightarrow A$ is $L = \cup_{\lambda \in U} L(\lambda)$, where*

$$L(\lambda) = d \oplus B(\lambda).$$

Proof. Using the identification as in Equations (28) and (29), we need to compute the image $(\theta^\# + \tau^\#)(0,\, h\,;\, X_- + k,\, X_+ - k)$ at each $\lambda \in U$. Now

$$\begin{aligned} \theta^\#(0,\, h\,;\, X_- + k,\, X_+ - k) &= (k,\, 0\,;\, h,\, h), \quad \text{and} \\ \tau^\#(0,\, h\,;\, X_- + k,\, X_+ - k) &= \frac{1}{2}(0,\, 0\,;\, \tau^\# X_+,\, \tau^\# X_+) - \frac{1}{2}(0,\, 0\,;\, \tau^\# X_-,\, \tau^\# X_-). \end{aligned}$$

Therefore,

$$\begin{aligned} & (\theta^\# + \tau^\#)(0,\, h\,;\, X_- + k,\, X_+ - k) + (0,\, h\,;\, X_- + k,\, X_+ - k) \\ = \;& \theta^\#(0,\, h\,;\, k,\, -k) + (0,\, h\,;\, k,\, -k) \\ & + \tau^\#(0,\, 0\,;\, X_-,\, X_+) + (0,\, 0\,;\, X_-,\, X_+). \end{aligned}$$

It is easy to see that

$$\theta^\#(0,\, h\,;\, k,\, -k) + (0,\, h\,;\, k,\, -k) = (k,\, h\,;\, h+k,\, h-k) \in d.$$

And

$$\begin{aligned} & \tau^\#(0,\, 0\,;\, X_-,\, X_+) + (0,\, 0\,;\, X_-,\, X_+) \\ = \;& \frac{1}{2}(0,\, 0\,;\, \tau^\# X_+,\, (\tau^\# + 2)X_+) - \frac{1}{2}(0,\, 0\,;\, (\tau^\# - 2)X_-,\, \tau^\# X_-) \\ = \;& \frac{1}{2}(0,\, 0\,;\, (r^\# - 1)X_+,\, (r^\# + 1)X_+) - \frac{1}{2}(0,\, 0\,;\, (r^\# - 1)X_-,\, (r^\# + 1)X_-) \in B(\lambda), \end{aligned}$$

where we have used the fact that $r^\#|_{\mathfrak{n}_\pm} = \tau^\# \pm 1$. This concludes the proof of the lemma.

$\square$

For any $\lambda \in U$, consider the decomposition:

$$\mathfrak{n}_{\pm} = \mathfrak{k}_{\pm}(\lambda) \oplus \mathfrak{n}_{\pm}^{\circ}(\lambda), \tag{34}$$

where

$$\mathfrak{k}_{\pm}(\lambda) = ker\tau^{\#}(\lambda) \cap \mathfrak{n}_{\pm} = span_{\mathbb{C}}\{E_{\pm\alpha} \,|\tau_{\alpha}(\lambda) = 0, \alpha \in \Delta_{+}\}, \tag{35}$$

and

$$\mathfrak{n}_{\pm}^{\circ}(\lambda) = span_{\mathbb{C}}\{E_{\pm\alpha} \,|\, \tau_{\alpha}(\lambda) \neq 0, \alpha \in \Delta_{+}\}. \tag{36}$$

Then we can rewrite $B(\lambda)$ as follows:

$$B(\lambda) = span_{\mathbb{C}}\{(0,\, 0\,;\, X,\, \varphi(\lambda)X), (0,\, 0\,;\, Y_{-},\, Y_{+}) \,|\, \forall X \in \mathfrak{n}_{\pm}^{\circ}(\lambda), Y_{\pm} \in \mathfrak{k}_{\pm}(\lambda)\}, \tag{37}$$

where

$$\varphi(\lambda) = \frac{r^{\#}(\lambda) + 1}{r^{\#}(\lambda) - 1} : \quad \mathfrak{n}_{\pm}^{\circ}(\lambda) \longrightarrow \mathfrak{n}_{\pm}^{\circ}(\lambda)$$

is the Cayley transformation of the linear operator $r^{\#}(\lambda)\,|_{\mathfrak{n}_{\pm}^{\circ}(\lambda)}$. Consequently, L can be written as:

$$\begin{aligned} L(\lambda) \;\; = \;\; & span_{\mathbb{C}}\{(k,\, 0\,;\, k,\, -k), (0,\, h\,;\, h,\, h), (0,\, 0\,;\, X,\, \varphi(\lambda)X), (0,\, 0\,;\, Y_{-},\, Y_{+}) \,| \\ & \forall h, k \in \mathfrak{h}, X \in \mathfrak{n}_{\pm}^{\circ}(\lambda), Y_{\pm} \in k_{\pm}(\lambda)\}. \end{aligned} \tag{38}$$

Lemma 4.2 *Assume that $L \subset E$ is a Dirac structure, then*

1. *both $\mathfrak{k}_{\pm}(\lambda)$ and $\mathfrak{n}_{\pm}^{\circ}(\lambda)$ are independent of $\lambda \in U$ (for simplicity, we denote them by $\mathfrak{k}_{\pm}$ and $\mathfrak{n}_{\pm}^{\circ}$ respectively);*

2. *$\mathfrak{n}_{\pm}^{\circ}$ are subalgebras of $\mathfrak{n}_{\pm}$;*

3. *$\mathfrak{k}_{\pm}$ are ideals of $\mathfrak{n}_{\pm}$.*

Proof. According to Theorem 3.1, $r_0 + \tau$ is a dynamical r-matrix. By Equation (26), we have

$$\begin{aligned} 0 \;\; &= \;\; [r_0, \tau] + \frac{1}{2}[\tau, \tau] + Alt(d\tau) \\ &= \;\; \sum_{i} \frac{\partial \tau}{\partial \lambda_i} \wedge h_i + \sum_{\alpha,\beta \in \Delta_{+}} [(\frac{1}{2}\tau_{\alpha} + 1)E_{\alpha} \wedge E_{-\alpha}, \;\; \tau_{\beta} E_{\beta} \wedge E_{-\beta}] \\ &= \;\; \sum_{\alpha \in \Delta_{+}} \sum_{i} \frac{\partial \tau_{\alpha}}{\partial \lambda_i} E_{\alpha} \wedge E_{-\alpha} \wedge h_i \;\; + \sum_{\alpha,\beta \in \Delta_{+}} (\frac{1}{2}\tau_{\alpha} + 1)\tau_{\beta}[E_{\alpha} \wedge E_{-\alpha}, \;E_{\beta} \wedge E_{-\beta}]. \end{aligned}$$

Since $[E_{\alpha},\, E_{-\alpha}] = h_{\alpha} = \sum < \alpha, h_i^{*} > h_i$ for any $\alpha \in \Delta_{+}$, the coefficient of the term $E_{\alpha} \wedge E_{-\alpha} \wedge h_i$ in the above equation is $\frac{\partial \tau_{\alpha}}{\partial \lambda_i} - < \alpha, h_i^{*} > (\tau_{\alpha} + 2)\tau_{\alpha}$. This implies that τ_{α} satisfies the following system of first-order differential equations:

$$\frac{\partial \tau_{\alpha}}{\partial \lambda_i} - < \alpha, h_i^{*} > (\tau_{\alpha} + 2)\tau_{\alpha} = 0, \quad \forall \alpha \in \Delta_{+}, \, (i = 1, \cdots, n).$$

Thus if $\tau_{\alpha}(\lambda_0) = 0$ for some $\lambda_0 \in U$, then $\tau_{\alpha} \equiv 0$ on U. This is equivalent to that $\mathfrak{k}_{\pm}(\lambda) = ker\tau(\lambda) \cap \mathfrak{n}_{\pm}$ are independent of $\lambda \in U$. This proves the first statement.

For the second statement, note that since $r(\lambda)$ is $\mathfrak{h}$-invariant, $\varphi(\lambda)$ commutes with $ad_{\mathfrak{h}}$. Thus $\varphi E_\alpha = \varphi_\alpha E_\alpha$ for some function $\varphi_\alpha : U \longrightarrow \mathbb{C}$, $\forall \alpha \in \mathfrak{n}^\circ{}_+$. For any $\alpha, \beta \in \mathfrak{n}^\circ{}_+$, since $(0,\, 0\,;\, E_\alpha,\, \varphi_\alpha E_\alpha), (0,\, 0\,;\, E_\beta,\, \varphi_\beta$ $\Gamma(L)$, their commutator belongs to $\Gamma(L)$ as well.

On the other hand, it is clear that

$$\begin{aligned} [(0,\, 0\,;\, E_\alpha,\, \varphi_\alpha E_\alpha), (0,\, 0\,;\, E_\beta,\, \varphi_\beta E_\beta)] &= (0,\, 0\,;\, [E_\alpha, E_\beta],\, \varphi_\alpha\varphi_\beta[E_\alpha, E_\beta]) \\ &\quad +(0,\, \frac{1}{4}(\varphi_\beta d\varphi_\alpha - \varphi_\alpha d\varphi_\beta) < E_\alpha, E_\beta >\,;\, 0,\, 0) \\ &= N_{\alpha,\beta}(0,\, 0\,;\, E_{\alpha+\beta},\, \varphi_\alpha\varphi_\beta E_{\alpha+\beta}). \end{aligned}$$

Here, in the last equality, we used the fact that $< E_\alpha, E_\beta >= 0$ whenever $\alpha \neq \beta$. According to Equation (37), we conclude that $E_{\alpha+\beta} \in \mathfrak{n}^\circ{}_+$ whenever $N_{\alpha,\beta} \neq 0$, i.e, $\alpha + \beta \in \Delta_+$. This means that $\mathfrak{n}^\circ{}_+$ is a Lie subalgebra of $\mathfrak{n}_+$ and

$$\varphi_\alpha\varphi_\beta \,=\, \varphi_{\alpha+\beta}, \qquad \forall E_\alpha, E_\beta \in \mathfrak{n}^\circ{}_+ \quad \text{such that} \quad \alpha + \beta \in \Delta_+. \tag{39}$$

Similarly we can prove that $\mathfrak{n}^\circ{}_-$ is a Lie subalgebra of $\mathfrak{n}_-$.

For the third statement, let $X_+, Y_+ \in \mathfrak{k}_+$, and $E_\alpha \in \mathfrak{n}^\circ{}_+$. As constant sections of $\Gamma(L)$, we have

$$[(0,\, 0\,;\, 0,\, X_+),\, (0,\, 0\,;\, 0,\, Y_+)] \,=\, (0,\, 0\,;\, 0,\, [X_+, Y_+]) \,\in \Gamma(L),$$

which means that $[X_+, Y_+] \in \mathfrak{k}_+$. Moreover,

$$[(0,\, 0\,;\, E_\alpha,\, \varphi_\alpha E_\alpha), (0,\, 0\,;\, 0,\, Y_+)] = (0,\, 0\,;\, 0,\, \varphi_\alpha[E_\alpha, Y_+]) \in \Gamma(L).$$

This implies that $[E_\alpha, Y_+] \in \mathfrak{k}_+$. Thus $\mathfrak{k}_+$ is an ideal of $\mathfrak{n}_+$ since $\mathfrak{n}_+ = \mathfrak{k}_+ \oplus \mathfrak{n}^\circ{}_+$. Similarly, $\mathfrak{k}_-$ is an ideal of $\mathfrak{n}_-$.

□

Below we will see that any decomposition $\mathfrak{n}_\pm = \mathfrak{k}_\pm \oplus \mathfrak{n}^\circ_\pm$ satisfying Properties (2)-(3) in Lemma 4.2 corresponds to a subset S of simple roots. More precisely, given a decomposition $\mathfrak{n}_\pm = \mathfrak{k}_\pm \oplus \mathfrak{n}^\circ_\pm$, let S be the subset of those simple roots α_i such that $E_{\alpha_i} \in \mathfrak{n}^\circ_+$. Define a subset of positive roots as follows:

$$[S] = \{\alpha \in \Delta_+ \mid \alpha = \sum_{\alpha_i \in S} n_i\alpha_i,\; n_i \geq 0\,\}. \tag{40}$$

Since any positive (negative) root can be expressed as positive (negative) linear combination of simple roots, we have

Proposition 4.3 *Assume that $\mathfrak{n}_\pm = \mathfrak{k}_\pm \oplus \mathfrak{n}^\circ_\pm$ is a decomposition satisfying Properties (2)-(3) in Lemma 4.2. Then,*

$$\mathfrak{n}^\circ{}_\pm = span_{\mathbb{C}}\{E_{\pm\alpha},\, \alpha \in [S]\}, \tag{41}$$

i.e., $\{E_{\pm\alpha_i} | \alpha_i \in S\}$ are Lie algebraic generators of $\mathfrak{n}^\circ{}_\pm$. Consequently,

$$\mathfrak{k}_\pm = span_{\mathbb{C}}\{E_{\pm\alpha},\, \alpha \in \Delta_+ \backslash [S]\}. \tag{42}$$

Conversely, given any subset S of simple roots, the corresponding $\mathfrak{k}_\pm$ and $\mathfrak{n}^\circ{}_\pm$ defined by Equations (41) and (42) above satisfy Properties (2)-(3) in Lemma 4.2.

Now we are ready to prove the main theorem of this section.

Theorem 4.4 *Let S be a subset of simple roots with corresponding $\mathfrak{k}_\pm$ and $\mathfrak{n}^\circ{}_\pm$ defined as in Proposition 4.3, and L a subbundle of E defined by Equation (38), where $\varphi(\lambda), \forall \lambda \in U$, is a linear operator on $n^\circ_\pm$. Then L is a Dirac structure iff there exists some $\lambda_0 \in \mathfrak{h}$ such that $\varphi(\lambda) = Ad_{e^{2(\lambda+\lambda_0)}}$.*

Proof. We shall divide the proof into four steps.

Step 1. It follows from Equations (30), (31) and (32) that for any $h \in \mathfrak{h}$ and $X \in \mathfrak{n}^\circ{}_\pm$,

$$[(0,\, h\,;\, h,\, h),\, (0,\, 0\,;\, X,\, \varphi X)\,] = (0,\, 0\,;\, [h, X],\, [h, \varphi X]).$$

It is still in L iff

$$[h, \varphi X] = \varphi[h, X],$$

which is equivalent to that φ commutes with $ad_{\mathfrak{h}}$. Therefore, $\varphi E_\alpha = \varphi_\alpha E_\alpha$, $\forall \alpha \in \pm[S]$, i.e., $E_\alpha \in n^\circ_\pm$, where φ_α is a complex valued function on U.

Step 2. Suppose that φ commutes with $ad_{\mathfrak{h}}$. Then $\forall i = 1, \cdots, n$ and $E_\alpha \in \mathfrak{n}^\circ{}_\pm$, both $(h_i^*,\, 0\,;\, h_i^*,\, -h_i^*)$ and $(0,\, 0\,;\, E_\alpha,\, \varphi_\alpha E_\alpha)$ are sections of L. By Equations (30), (31) and (32)

$$\begin{aligned}
[(h_i^*,\, 0\,;\, h_i^*,\, -h_i^*), (0,\, 0\,;\, E_\alpha,\, \varphi_\alpha E_\alpha)] &= (0,\, 0\,;\, 0,\, \frac{\partial \varphi_\alpha}{\partial \lambda_i} E_\alpha) + (0,\, 0\,;\, [h_i^*, E_\alpha],\, -\varphi_\alpha [h_i^*, E_\alpha]) \\
&= (0,\, 0\,; <\alpha, h_i^*> E_\alpha,\, (\frac{\partial \varphi_\alpha}{\partial \lambda_i} - <\alpha, h_i^*> \varphi_\alpha) E_\alpha).
\end{aligned}$$

It is still in $\Gamma(L)$ iff

$$\frac{\partial \varphi_\alpha}{\partial \lambda_i} = 2 <\alpha, h_i^*> \varphi_\alpha \qquad \Longleftrightarrow \qquad \varphi_\alpha(\lambda) = C_\alpha e^{2<\alpha,\lambda>},$$

where C_α are certain constants and $\lambda = \sum \lambda_i h_i^*$.

Step 3. Suppose that $\varphi_\alpha(\lambda) = C_\alpha e^{2<\alpha,\lambda>}$. Next we show that C_α satisfy the following relations:

$$C_{-\alpha} = C_\alpha^{-1}, \qquad C_{\alpha+\beta} = C_\alpha C_\beta, \qquad \forall \alpha, \beta \in \pm[S], \quad \text{whenever } \alpha + \beta \in \Delta. \tag{43}$$

When $\beta \neq -\alpha$ the conclusion follows from Equation (39), where only a special case: both α and β being positive roots, is discussed. However, the general situation can also be easily checked using the fact that $< E_\alpha, E_\beta >= 0$. Now assume that $\beta = -\alpha$. Then, by Equation (31), we have

$$\begin{aligned}
& [(0, 0; E_\alpha, \varphi_\alpha E_\alpha), (0,\, 0\,;\, E_{-\alpha},\, \varphi_{-\alpha} E_{-\alpha})] \\
= \;& [(0, 0; E_\alpha,\, e^{2<\alpha,\lambda>} C_\alpha E_\alpha),\, (0,\, 0\,;\, E_{-\alpha},\, e^{-2<\alpha,\lambda>} C_{-\alpha} E_{-\alpha})] \\
= \;& (0,\, \frac{1}{4}(e^{-2<\alpha,\lambda>} d e^{2<\alpha,\lambda>} - e^{2<\alpha,\lambda>} d e^{-2<\alpha,\lambda>}) < E_\alpha, E_{-\alpha} >;\, [E_\alpha, E_{-\alpha}],\, C_\alpha C_{-\alpha} [E_\alpha, E_{-\alpha}]) \\
= \;& (0,\, \sum_i \frac{\partial <\alpha, \lambda>}{\partial \lambda_i} d\lambda_i\,;\, [E_\alpha, E_{-\alpha}],\, C_\alpha C_{-\alpha} [E_\alpha, E_{-\alpha}]) \\
= \;& (0,\, \sum <\alpha, h_i^*> h_i\,;\, h_\alpha,\, C_\alpha C_{-\alpha} h_\alpha) \\
= \;& (0,\, h_\alpha\,;\, h_\alpha,\, C_\alpha C_{-\alpha} h_\alpha),
\end{aligned}$$

where we used the facts that

$$d\lambda_i = h_i, \quad \lambda = \sum \lambda_i h_i^*, \quad h_\alpha = \sum < \alpha, h_i^* > h_i \quad and \quad < E_\alpha, E_{-\alpha} >= 1. \tag{44}$$

Obviously the commutator is still in $\Gamma(L)$ iff $C_\alpha C_{-\alpha} = 1$. Thus, Equation (43) is proved.

Finally, it is not difficult to see that Equation (43) implies that there exists some $\lambda_0 \in \mathfrak{h}$ such that $C_\alpha = e^{2<\alpha,\lambda_0>}$. In fact, we can take $\lambda_0 = \frac{1}{2}\sum_{\alpha_i \in S}(ln\, C_{\alpha_i})h_i^*$. Consequently, we have

$$\varphi_\alpha(\lambda) = e^{2<\alpha,\lambda+\lambda_0>} \qquad \Longleftrightarrow \qquad \varphi(\lambda) = Ad_{e^{2(\lambda+\lambda_0)}}. \tag{45}$$

Conversely, if $\varphi(\lambda) = Ad_{e^{2(\lambda+\lambda_0)}}$, L is maximal isotropic since φ preserves the Killing form $< \cdot, \cdot >$. Moreover, $\Gamma(L)$ is closed, so L is indeed a Dirac structure. This concludes the proof.

□

Corollary 4.5 *([10]) A meomorphic function $r : U \longrightarrow \wedge^2\mathfrak{g}$ is a dynamical r-matrix iff r is of the form:*

$$r(\lambda) = \omega + \sum_{\alpha\in[S]} \coth < \alpha, \lambda + \lambda_0 > E_\alpha \wedge E_{-\alpha} + \sum_{\alpha\in\Delta_+\setminus[S]} E_\alpha \wedge E_{-\alpha}, \tag{46}$$

where ω is a closed 2-form on U, and $[S]$ is defined by Equation (40) for a subset S of the simple roots.

Proof. Let $\tau = r - r_0$. Then τ is of the form:

$$\tau = \omega + \sum_{\alpha\in\Delta_+} \tau_\alpha E_\alpha \wedge E_{-\alpha}, \tag{47}$$

where ω is a closed two-form on U.

According to Theorem 3.1, r is a dynamical r-matrix iff $\Gamma_{\theta+\tau} \subset A \oplus A^*$ is a Dirac structure of the Lie bialgebroid (A, A^*, r_0). Without loss of generality, assume that $\omega = 0$. According to Theorem 4.4, the latter amounts to that there exists a subset of simple roots S with corresponding $\mathfrak{n}^\circ_\pm$ and $\mathfrak{k}_\pm$ such that the Cayley transformation of $r^\#|_{\mathfrak{n}^\circ{}_\pm}$: $\varphi(\lambda) = \frac{r_1^\#(\lambda)+1}{r_1^\#(\lambda)-1}$ has expression (45), for some fixed $\lambda_0 \in \mathfrak{h}$. This immediately implies that

$$r(\lambda)|_{\mathfrak{n}^\circ{}_\pm} = \sum_{\alpha\in[S]} coth < \alpha, \lambda + \lambda_0 > E_\alpha \wedge E_{-\alpha}, \qquad \text{and} \qquad r(\lambda)|_{\mathfrak{k}_\pm} = \sum_{\alpha\in\Delta_+\setminus[S]} E_\alpha \wedge E_{-\alpha}.$$

The conclusion thus follows.

□

5 Lagrangian subalgebras and dynamical r-matrices

In [14], Karolinsky classified all Lagrangian subalgebras W_0 of the double of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, r_0)$ (by abuse of notation, in the sequel, we will simply say Lagrangian subalgebras of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, r_0)$) in terms of the triples $(\mathfrak{u}^-, \mathfrak{u}^+, \varphi)$, where $\mathfrak{u}^\pm$ are two parabolic subalgebras of $\mathfrak{g}$, $\mathfrak{m} = \mathfrak{u}^+ \cap \mathfrak{u}^-$ is a Levi subalgebra and φ is an inner automorphism of $\mathfrak{m}$. The following theorem shows that such a classification can be reduced to a simpler form in the special case that $W_0 \cap \mathfrak{g} = \mathfrak{h}$.

Proposition 5.1 *There is a one-one correspondence between Lagrangian subalgebras $W_0 \subset \mathfrak{g} \oplus \mathfrak{g}$ with $W_0 \cap \mathfrak{g} = \mathfrak{h}$ and pairs (S, λ_0), where S is a subset of simple roots and $\lambda_0 \in \mathfrak{h}^*$.*

Proof. Given such a pair (S, λ_0), define $\mathfrak{n}^\circ_\pm$ and $\mathfrak{k}_\pm$ as in Proposition 4.3 by:

$$\mathfrak{n}^\circ{}_\pm = span_{\mathbb{C}}\{E_{\pm\alpha},\ \alpha \in [S]\}, \tag{48}$$

$$\mathfrak{k}_\pm = span_{\mathbb{C}}\{E_{\pm\alpha},\ \alpha \in \Delta_+ \backslash [S]\}, \tag{49}$$

where

$$[S] = \{\alpha \in \Delta_+ \mid \alpha = \sum_{\alpha_i \in S} n_i\alpha_i,\ n_i \geq 0\,\}. \tag{50}$$

Let $W_0 \subset \mathfrak{g} \oplus \mathfrak{g}$ be the subspace:

$$W_0 = span_{\mathbb{C}}\{(h,h),\ (X, Ad_{e^{\lambda_0}}X),\ (Y_-, Y_+)\ \mid \forall h \in \mathfrak{h},\ X \in \mathfrak{n}^\circ_\pm,\ Y_\pm \in \mathfrak{k}_\pm\}. \tag{51}$$

One can check directly that W_0 is a Lagrangian subalgebra and $W_0 \cap \mathfrak{g} = \mathfrak{h}$. Here, as before, the double is identified with $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, whereas $\mathfrak{g}$ is identified with the diagonal of $\mathfrak{d}$.

Conversely, as we know in Section 2, any Lagrangian subalgebra of the double of a Lie bialgebra arises from a characteristic pair. More precisely, given a Lagrangian subalgebra $W_0 \subset \mathfrak{g} \oplus \mathfrak{g}$ such that $W_0 \cap \mathfrak{g} = \mathfrak{h}$, there exists some $J \in \mathfrak{g} \wedge \mathfrak{g}$ such that

$$W_0 = \{\, X + J^{\#}\xi + \,\xi \,|\, X \in \mathfrak{h}, \xi \in \mathfrak{h}^\perp\} = \mathfrak{h} \oplus graph(J^{\#}|_{\mathfrak{h}^\perp}), \tag{52}$$

i.e., $(\mathfrak{h}, J)$ is a characteristic pair of W_0. However J is not unique. What we need here is to choose an $\mathfrak{h}$-invariant J. For this purpose, we notice that $\forall X \in \mathfrak{h}$ and $\xi \in \mathfrak{h}^\perp$,

$$\begin{aligned}
[X, J^{\#}\xi + \xi] &= [X, J^{\#}\xi] + [X, \xi] \\
&= [X, J^{\#}\xi] + ad^*_X\xi - ad^*_\xi X \\
&= \{[X, J^{\#}\xi] - J^{\#}(ad^*_X\xi)\} \ + \ \{J^{\#}(ad^*_X\xi) + ad^*_X\xi\}.
\end{aligned}$$

Here we used the fact that $ad^*_\xi X = 0$, which can be easily verified directly. It is easy to see that $ad^*_X\xi \in \mathfrak{h}^\perp$ so that $J^{\#}(ad^*_X\xi) + ad^*_X\xi \in W_0$. Thus, $[X, J^{\#}\xi + \xi] \in W_0$ iff

$$[X, J^{\#}\xi] - J^{\#}(ad^*_X\xi) = (ad_X \circ J^{\#} - J^{\#} \circ ad^*_X)\xi = [X, J]^{\#}\xi \ \in \mathfrak{h}.$$

Equivalently,

$$[X,\ J] \equiv 0\,(mod\,\mathfrak{h}),\quad \forall X \in \mathfrak{h}, \tag{53}$$

i.e., J is $ad_{\mathfrak{h}}$ - invariant (mod $\mathfrak{h}$). Notice that, as an element of $\mathfrak{g} \wedge \mathfrak{g}$, J can always be written as:

$$J = \sum_{\alpha,\beta \in \Delta} J_{\alpha,\beta} E_\alpha \wedge E_\beta \ + J_1$$

where $J_1 \equiv 0$ (mod $\mathfrak{h}$). In fact, one can always take $J_1 = 0$, which will not affect the Lagrangian subalgebra W_0. Moreover, it follows from the equation:

$$[h, E_\alpha \wedge E_\beta] = [h, E_\alpha] \wedge E_\beta + E_\alpha \wedge [h, E_\beta] =< \alpha + \beta, h > E_\alpha \wedge E_\beta, \quad \forall h \in \mathfrak{h},$$

that $J_{\alpha,\beta} = 0$ whenever $\alpha + \beta \neq 0$. By denoting $J_{\alpha,-\alpha}$ by J_α, we can write

$$J = \sum_{\alpha \in \Delta_+} J_\alpha E_\alpha \wedge E_{-\alpha}, \tag{54}$$

which is in fact $ad_{\mathfrak{h}}$ - invariant. Thus, under the standard identification that $\mathfrak{g} \oplus \mathfrak{g}^* \cong \mathfrak{d}(= \mathfrak{g} \oplus \mathfrak{g})$, W_0 is of the form (comparing with Equation (37) in the last section):

$$W_0 = \{(h,h),\ (X, \varphi X),\ (Y_-, Y_+)\ \mid h \in \mathfrak{h},\ X \in \mathfrak{n}^\circ_\pm,\ Y_\pm \in \mathfrak{k}_\pm\}, \tag{55}$$

where $\mathfrak{k}_\pm = span_{\mathbb{C}}\{E_{\pm\alpha}\,|J_\alpha = 0, \alpha \in \Delta_+\}$, $\mathfrak{n}^\circ_\pm = span_{\mathbb{C}}\{E_{\pm\alpha}\,|\,J_\alpha \neq 0, \alpha \in \Delta_+\}$ in analogous to Equation (35), and φ is the Cayley transformation of $(J^\# + r_0^\#)|_{\mathfrak{n}^\circ_\pm}$: $\varphi(\lambda) = \frac{(J^\# + r_0^\#)+1}{(J^\# + r_0^\#)-1}$. Using a similar argument as in the proof of Lemma 4.2, we can show that $\mathfrak{n}^\circ_\pm$ are indeed subalgebras of $\mathfrak{n}_\pm$ and $\mathfrak{k}_\pm$ are ideals of $\mathfrak{n}_\pm$. Consequently, they correspond to a subset S of the set of simple roots according to Proposition 4.3.

Finally, by using the fact that the commutator of the elements $(E_\alpha, \varphi_\alpha E_\alpha)$ is still in W_0, one derives the following relations:

$$\varphi_{-\alpha} = \varphi_\alpha^{-1}, \qquad \varphi_{\alpha+\beta} = \varphi_\alpha \varphi_\beta, \qquad \forall \alpha, \beta \in \pm[S] \quad \text{such that } \alpha + \beta \in \Delta.$$

This implies that $\varphi = Ad_{e^{\lambda_0}}$ for some $\lambda_0 \in \mathfrak{h}$. This concludes the proof.

□

In the sequel, we use $l(S, \lambda_0)$ to denote the Lagrangian subalgebra W_0 corresponding to the pair (S, λ_0). Combining the above proposition and Corollary 4.5 leads to:

Theorem 5.2 *There is a one-one correspondence among the following objects:*

1. *dynamical r-matrices with zero gauge term,*
2. *pairs (S, λ_0), where S is a subset of the simple roots and $\lambda_0 \in \mathfrak{h}^*$, and*
3. *Lagrangian subalgebras $W_0 \subset \mathfrak{g} \oplus \mathfrak{g}$ such that $W_0 \cap \mathfrak{g} = \mathfrak{h}$.*

This theorem establishes the correspondence between Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ and dynamical r-matrices in a rather indirect manner, namely through the pair (S, λ_0). Next we will illuminate a direct connection geometrically.

Consider a subbundle $U \times \mathfrak{h}$ of A, where $U \subset TU$ is identified with the zero section and $\mathfrak{h} \subset \mathfrak{g}$. Given a function $\tau : U \longrightarrow \wedge^2\mathfrak{g}$, being considered as a section in $\Gamma(\wedge^2 A)$, the characteristic pair $(U \times \mathfrak{h}, \tau)$ defines a maximal isotropic subbundle W of $A \oplus A^*$:

$$W = \{\, X + \tau^\# \xi + \,\xi \,|\, X \in U \times \mathfrak{h}, \xi \in (U \times \mathfrak{h})^\perp\}, \tag{56}$$

as given by Equation (16). Then we have

Proposition 5.3 *If $r(\lambda) = \tau(\lambda) + r_0$ is a dynamical r-matrix, the subbundle W corresponding to the characteristic pair $(U \times \mathfrak{h},\, \tau)$ is a Dirac structure of the Lie bialgebroid $(A, A^*,\, r_0)$.*

Proof. It suffices to check the three conditions in Theorem 2.4. First, it is obvious that $U \times \mathfrak{h} \subset A$ is a Lie subalgebroid. Second, we have

$$\begin{aligned} d_* \tau + \frac{1}{2}[\tau, \tau] &= [r_0, \tau] + \frac{1}{2}[\tau, \tau] \\ &= -\sum h_i \wedge \frac{\partial \tau}{\partial \lambda_i} \equiv 0 \ \ (mod U \times \mathfrak{h}), \end{aligned}$$

according to Equation (26).

Third, $\forall \xi, \eta \in \Gamma((U \times \mathfrak{h})^{\perp})$ and $h \in \mathfrak{h}$,

$$\begin{aligned} < L_{\tau^{\#}\xi}\eta, h > &= < \eta\,,\, [h, \tau^{\#}\xi] > \\ &= < \eta, \tau^{\#}(L_h \xi) > \\ &= < [h, \tau^{\#}\eta], \xi > - L_h < \tau^{\#}\eta, \xi > \\ &= < L_{\tau^{\#}\eta}\xi, h > + < d < \tau^{\#}\xi, \eta >, h >, \end{aligned}$$

where in the second equality we used the fact that τ is $\mathfrak{h}$-invariant. It thus follows that

$$< [\xi, \eta]_\tau, h > = < L_{\tau^{\#}\xi}\eta - L_{\tau^{\#}\eta}\xi - d\langle \tau^{\#}\xi, \eta \rangle, h > = 0, \qquad \forall h \in \mathfrak{h}.$$

That is, $\Gamma(U \times \mathfrak{h})^{\perp}$ is closed under $[\cdot, \cdot]_\tau$. On the other hand, it is well known that $\mathfrak{h}^{\perp}$ is an ideal of the dual Lie algebra $\mathfrak{g}^*$, since $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra. This means that $(U \times \mathfrak{h})^{\perp}$ is a Lie subalgebroid of A^*. Thus, $\Gamma(U \times \mathfrak{h})^{\perp}$ is closed under the bracket $[\xi, \eta] + [\xi, \eta]_\tau$. Consequently, the conclusion follows.

□

It is well known that a Lie bialgebra integrates to a Poisson group. Similarly, the global object corresponding to a Lie bialgebroid is a Poisson groupoid [22] [23]. For the Lie bialgebroid (A, A^*, r_0), its Poisson groupoid is very simple to describe. As a groupoid, it is simply the product of the pair groupoid $U \times U$ with the Lie group G, where G is a Lie group with Lie algebra $\mathfrak{g}$. The Poisson structure is the product of the zero Poisson structure on $U \times U$ with the Poisson group structure on G defined by the r-matrix r_0. According to Theorem 8.6 in [18], the Dirac structure W corresponds to a Poisson homogeneous space Q of this Poisson groupoid. As a manifold,

$$Q = (U \times U \times G)/(U \times H) \cong U \times G/H,$$

where $H \subset G$ is a closed subgroup with Lie algebra $\mathfrak{h}$. It is not difficult to see that for each fixed $\lambda \in U$, $\{\lambda\} \times G/H$ is a Poisson submanifold, whereas the Poisson tensor is

$$\pi_Q(\lambda) = p_*(r_0^L - r_0^R + \tau^L(\lambda)) = p_*(r^L(\lambda) - r_0^R).$$

Here $p : G \longrightarrow G/H$ is the projection, $r^L(\lambda)$ refers to the bivector field on G obtained by the left translation of $r(\lambda) \in \wedge^2\mathfrak{g}$, and r_0^R refers to the bivector field on G obtained by the right translation of $r_0 \in \wedge^2\mathfrak{g}$. It is simple to see that $(G/H, \pi_Q(\lambda))$ is a Poisson homogeneous G-space. Thus in this way we obtain a family of Poisson homogeneous G-spaces parameterized by $\lambda \in U$. It is not surprising that this is the family of Poisson homogeneous spaces studied by Lu [20].

The corresponding family of Lagrangian subalgebras (or Dirac structures) of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, r_0)$ is just the fibers of W:

$$W(\lambda) = \{\, X + \tau^{\#}(\lambda)\xi + \xi \,|\, X \in \mathfrak{h}, \xi \in \mathfrak{h}^{\perp}\}. \tag{57}$$

In other words, $W(\lambda)$ corresponds to the characteristic pair $(\mathfrak{h},\ \tau(\lambda))$. In fact, it is easy to see that

$$W(\lambda) = l(S, \lambda + \lambda_0), \tag{58}$$

where (S, λ_0) is the pair corresponding to the dynamical r-matrix $r(\lambda)$ as in Theorem 5.2. We now summarize the above discussion in the following two corollaries.

Corollary 5.4 *The following two statements are equivalent:*

1. *The subbundle W defined by the characteristic pair $(U \times \mathfrak{h}, \tau)$ is a Dirac structure of the Lie bialgebroid (A, A^*, r_0).*

2. *For any fixed $\lambda \in U$, $W(\lambda)$ defined by the characteristic pair $(\mathfrak{h}, \tau(\lambda))$ is a Dirac structure for the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, r_0)$.*

Corollary 5.5 *[20] A dynamical r-matrix $r(\lambda)$ defines a family of Dirac structures $W(\lambda)$ of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, r_0)$, which in turn corresponds to a family of Poisson homogeneous G -spaces $(G/H, \pi_Q(\lambda))$.*

Such a family of Lagrangian subalgebras is said to be governed by a dynamical r-matrix. From Corollary 5.4, we see that the inverse of Proposition 5.3 is not necessary true, because W being a Dirac structure is only a fiberwise property without involving any dynamical relation. In fact, given a family of Lagrangian subalgebras $W(\lambda)$, $\forall \lambda \in U$, we may write $W(\lambda) = l(S_\lambda,\ \psi(\lambda))$ for $\psi(\lambda) \in \mathfrak{h}^*$. From Equation (58), it follows that $W(\lambda)$ is governed by a dynamical r-matrix iff S_λ is independent of λ and $\psi : \mathfrak{h} \longrightarrow \mathfrak{h}$ is a linear translation: $\psi(\lambda) = \lambda + \lambda_0$ for some $\lambda_0 \in \mathfrak{h}$. Consequently, we have

Corollary 5.6 *Let $\mu \in U$ be any fixed point, W_0 a Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ such that $W_0 \cap \mathfrak{g} = \mathfrak{h}$. Then W_0 extends uniquely to a family of Lagrangian subalgebras $W(\lambda)$ such that $W(\mu) = W_0$, which is governed by a dynamical r-matrix.*

Proof. Assume that $W_0 = l(S, \lambda_0)$. Consider the pair $(S, \lambda_0 - \mu)$. This corresponds to a dynamical r- matrix $r(\lambda)$ according to Theorem 5.2. Let $W(\lambda)$ be its corresponding family of Lagrangian subalgebras. Then $W(\lambda) = l(S, \lambda - \mu + \lambda_0)$. Thus $W(\mu) = l(S, \lambda_0) = W_0$. Moreover, it is clear that such an extension is unique.

□

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